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HAUSDORFF MEASURE OF p-CANTOR SETS

Abstract

In this paper we analyze Cantor type sets constructed by the removal of open intervals whose lengths are the terms of the p-sequence, $\{k^{-p}\}_{k=1}^{\infty}$. We prove that these Cantor sets are s-sets, by providing sharp estimates of their Hausdorff measure and dimension.

Sets of similar structure arise when studying the set of extremal points of the boundaries of the so-called random stable zonotopes.

1 Introduction and Notation

By a general Cantor set, we mean a compact, perfect, totally disconnected subset of the real line. In this paper we will only consider Cantor sets of

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zero Lebesgue measure. Cantor sets arise in many different settings including as examples having many surprising properties. They can be constructed in different ways.

General Cantor sets can be constructed in a manner similar to that of the classical middle-third Cantor set. Let $\lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers, such that $\sum_k \lambda_k = K < +\infty$. We associate to such a sequence, the Cantor set C_λ in the following way. Start with the closed interval $I_0 = [0, K]$ of length, $|I_0| = K = \sum_{k=1}^{\infty} \lambda_k$. (Clearly by normalization we can always achieve $I_0 = [0, 1]$.)

In what follows, we use the notation $|I|$ for the length$(I) = \mathrm{diam}(I)$ of any interval. In the first step, remove from I_0 an open interval of length λ_1, obtaining the two closed intervals of step 1, I_0^1 and I_1^1, and a gap of length λ_1 between them. Shortly we will see that the location of the gap to be removed is uniquely determined.

Having completed step k, we will have 2^k closed intervals I_ℓ^k, $\ell = 0, \dots, 2^k - 1$. From each of them we remove an open interval of length equal to the next unused term of the sequence, thus, from I_ℓ^k, we remove an open interval of length $\lambda_{2^k+\ell}$. Again, we will see that the location of the gap is uniquely determined. This forms in I_ℓ^k, two closed sub-intervals $I_{2\ell}^{k+1}$ and $I_{2\ell+1}^{k+1}$, with respect to which the following relation holds,

$$|I_\ell^k| = |I_{2\ell}^{k+1}| + \lambda_{2^k+\ell} + |I_{2\ell+1}^{k+1}|.$$

As noted above, in order for this construction to be possible, the position of the gaps removed at each step is not arbitrary. Since the length of the interval I_0 equals the sum of the lengths of **all** the intervals removed in the construction, there is a unique way of doing this construction. That is, the length of each of the remaining intervals at step k should be exactly the sum of the lengths of all the gaps that will be removed from it later in the construction. So, the two remaining intervals of step 1 will have lengths:

$$|I_0^1| = \sum_{n=1}^{\infty} \sum_{j=0}^{2^{n-1}-1} \lambda_{2^n+j}, \qquad |I_1^1| = \sum_{n=1}^{\infty} \sum_{j=2^{n-1}}^{2^n-1} \lambda_{2^n+j}.$$

In general, for all $k = 0, 1, \dots$ and $\ell = 0, 1, \dots, 2^k - 1$, we have

$$|I_\ell^k| = \sum_{n=k}^{\infty} \sum_{j=\ell 2^{n-k}}^{(\ell+1)2^{n-k}-1} \lambda_{2^n+j}. \tag{1}$$

We denote by C_λ^k the union of the closed intervals at the kth step

$$C_\lambda^k = \bigcup_{\ell=0}^{2^k-1} I_\ell^k.$$

Then

$$C_\lambda = \bigcap_{k=0}^{\infty} C_\lambda^k.$$

We will say that the Cantor set constructed in this way using a sequence λ, is the *Cantor set associated to the sequence* λ.

Remark. This construction is quite general since in fact any Cantor set (of zero Lebesgue measure) can be obtained in this way for an appropriate choice of the sequence. Namely, let C be a Cantor set in $\mathbb{R}$ and let I_0 be the smallest interval containing C. The complement of C in I_0 is a countable union of open intervals U_i, such that $\sum_{i\in\mathbb{N}} |U_i| = |I_0|$. The following procedure will show how to define a sequence $a = \{a_k\}$, such that $C_a = C$.

Let U_{i_1} be a gap of maximal length and define $a_1 = |U_{i_1}|$. Next choose U_{i_2} a gap of maximal length to the left of U_{i_1}, and U_{i_3} a gap of maximal length to the right of U_{i_1}. Now define $a_2 = |U_{i_2}|$ and $a_3 = |U_{i_3}|$. In the next step define a_4 through a_7 by picking a gap of maximal length in each of the remaining intervals (i.e. $I_0 - (U_{i_1} \cup U_{i_2} \cup U_{i_3})$.) Continuing in this fashion, the sequence $a = \{a_k\}$ satisfies $C_a = C$.

It is clear from the construction of the Cantor set associated to a given sequence, that the specific order in which the gaps appear in the sequence determines the resulting Cantor set. Of central importance in this paper is the investigation of the effect on Hausdorff dimension (defined below) due to rearrangements of the sequence of gaps.

Definition. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijective map; we say that the sequence $\{\lambda_{\sigma(k)}\}_{k\in\mathbb{N}}$ is a *rearrangement* of λ and denote it by $\sigma(\lambda)$.

Remark. In general, a rearrangement of the original sequence yields a different Cantor set. As we will see in this paper, the new Cantor set can have a different Hausdorff dimension than the original. It is also possible for the new Cantor set to be the same as the original. To see this, repeat the construction in the Remark above, but now make a different choice of gaps at each level other than the one with maximal diameter. The only requirement is that at some step, each gap is eventually selected. On the other hand, if two different sequences yield the same Cantor set, evidently one is a rearrangement of the other.

We recall the definitions of Hausdorff measure and dimension.

Definition. Let $A \subset \mathbb{R}$ be a Borel-measurable set and $\alpha > 0$. For $\delta > 0$ let

$$\mathcal{H}_\delta^\alpha(A) = \inf\Big\{ \sum (\mathrm{diam}(E_i))^\alpha \ : \ E_i \text{ open}, \ \cup E_i \supset A, \mathrm{diam}(E_i) \le \delta \Big\}.$$

Then, the *α-dimensional Hausdorff measure of A*, $\mathcal{H}^\alpha(A)$, is defined as

$$\mathcal{H}^\alpha(A) = \lim_{\delta\to 0} \mathcal{H}_\delta^\alpha(A),$$

and the Hausdorff dimension of A is,

$$\dim_H(A) = \sup\{\alpha : \mathcal{H}^\alpha(A) > 0\}.$$

It can be shown ([6]), that if in the definition of the Hausdorff measure, the elements of the coverings are chosen to be closed sets, or Borel sets, the resulting measure is the same.

If for some choice $\alpha = s$, $0 < \mathcal{H}^s(A) < \infty$, then A is called an s-set. Since we will only be using the Hausdorff dimension in this paper, henceforth we omit the subscript H.

We will mainly consider Cantor sets constructed using the p-sequence $\lambda = \{\lambda_k\}_{k\in\mathbb{N}}$, and its rearrangements, where $\lambda_k = k^{-p}$. We call such sets *p-Cantor sets*. Sets of similar structure arise in the analysis of the extremal points of boundaries of random stable zonotopes, which were studied in [2]. However such extremal sets are more complicated than the Cantor sets treated here since they are random and lie in $\mathbb{R}^d$ with $d \geq 2$.

In general, the computation of the Hausdorff dimension or the Hausdorff measure of a set is not easy to do, see [3], [5] and references therein. Estimates from above are usually simpler to obtain than estimates from below. In our particular case, showing that the Hausdorff measure is finite, for an appropriate choice of s, will be relatively easy. However, showing that it is positive will require sharp estimates on the size of the intervals of the construction.

We now state our main results.

Theorem 1.1 *Let $\lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ be defined by $\lambda_k = \left(\frac{1}{k}\right)^p$, $p > 1$. Then C_λ is a $\frac{1}{p}$-set, precisely,*

$$\frac{1}{8}\left(\frac{2^p}{2^p - 2}\right)^{\frac{1}{p}} \leq \mathcal{H}^{\frac{1}{p}}(C_\lambda) \leq \left(\frac{1}{p-1}\right)^{\frac{1}{p}}$$

and furthermore

$$dim\ C_\lambda = \frac{1}{p}.$$

The second theorem deals with rearrangements of the original sequence λ.

Theorem 1.2 *If $\sigma(\lambda)$ is any rearrangement of the sequence $\lambda = k^{-p}$, $k \in \mathbb{N}$, with $p > 1$, then*

$$0 \leq dim\, C_{\sigma(\lambda)} \leq \frac{1}{p}.$$

Furthermore, for each $0 < s \leq \frac{1}{p}$, there exists a rearrangement $\sigma_s(\lambda)$ such that $C_{\sigma_s(\lambda)}$ is an s-set.

We remark here that our main goal is to show that p-Cantor sets are s-sets for an appropriate choice of s. If we were only interested in determining the Hausdorff dimension of these sets, we could use the results in the article of Beardon [1] about the Hausdorff dimension of what is referred to there as general C-sets; these sets were introduced by Tsuji [7].

The result of Theorem 1.1 fits nicely into a result by Falconer [4] pg. 55, where he computes the Box dimension for Cantor sets constructed in this fashion. He shows that the upper and lower Box dimension coincide if and only if the following limit

$$\ell = \lim_{k\to\infty} \frac{\log \lambda_k}{\log k}$$

exists, in which case the Box dimension is $-1/\ell$. For the particular case of the p-series, this limit is $-p$, yielding yet another way for obtaining an upper bound for the Hausdorff dimension.

2 Cantor sets associated to geometrical sequences.

In this section, we will briefly leave the p-series, and analyze the behavior of Cantor sets associated to sequences with geometric decay.

Definition We say that a sequence of positive terms $a = \{a_k\}$ has *at least geometrical decay*, if there exist $0 < d < 1$ and $c > 0$ such that $a_k \leq cd^k$ for all $k \in \mathbb{N}$.

We will show next, that a sequence which tends to zero can be decomposed into finitely many or countably many subsequences, all of them having at least geometrical decay.

Lemma 2.1 *Let $a = \{a_n\}$ be a sequence of positive terms such that $\lim a_n = 0$. Then there exists a family of functions $\{\gamma_j : \mathbb{N} \to \mathbb{N},\quad j = 1, 2, \ldots\}$ at most countable such that*

1. *γ_j is one to one and increasing for all j.*
2. *$\gamma_j(\mathbb{N}) \bigcap \gamma_{j'}(\mathbb{N}) = \emptyset$ if $j \neq j'$.*
3. *$\mathbb{N} = \bigcup_j \gamma_j(\mathbb{N})$.*
4. *For all j, the subsequence $a^{(j)} = \{a_{\gamma_j(n)}\}_{n\in\mathbb{N}}$ has at least geometrical decay.*

Proof. We define first γ_1 by

$$\begin{aligned} \gamma_1(1) &= 1 \text{ and if } \gamma_1(n) \text{ is already defined then} \\ \gamma_1(n+1) &= \min\{m \in \mathbb{N} : m > \gamma_1(n) \text{ and } a_m \leq 1/2^{n+1}\}; \end{aligned}$$

this being possible since $a_n \longrightarrow 0$. This defines γ_1 inductively. Now we assume that $\gamma_1, ..., \gamma_k$ are already defined, then if $\mathbb{N} \setminus \bigcup_{j=1}^k \gamma_j(\mathbb{N})$ is finite, we stop, and redefine γ_1 in such a way that $\gamma_1(\mathbb{N}) = \mathbb{N} \setminus \bigcup_{j=2}^k \gamma_j(\mathbb{N})$. Otherwise, we define γ_{k+1} by

$$\begin{aligned} \gamma_{k+1}(1) &= \min(\mathbb{N} \setminus \bigcup_{j=1}^{k} \gamma_j(\mathbb{N})) \text{ and if } \gamma_{k+1}(n) \text{ is already defined then} \\ \gamma_{k+1}(n+1) &= \min\{m \in \mathbb{N} \setminus (\bigcup_{j=1}^{k} \gamma_j(\mathbb{N})) : m > \gamma_{k+1}(n) \text{ and } a_m \leq 1/2^{n+1}\}. \end{aligned}$$

If the process does not end in a finite number of steps then $\mathbb{N} = \bigcup_j \gamma_j(\mathbb{N})$ since every number n must be selected at most at step n. ∎

Let us now prove, that a Cantor set associated to a sequence with at least geometric decay, has Hausdorff dimension 0.

Proposition 2.2 *Let $a = \{a_k\}_{k \in \mathbb{N}}$ be any sequence such that $0 < a_k \leq r^k$ for $r < 1$. Then, the Cantor set C_a has Hausdorff dimension 0.*

Proof. We will show, that for each $\epsilon > 0$, $\dim C_a \leq \epsilon$. Suppose that n is sufficiently large and such that $\sum_{j=n+1}^{\infty} r^j \leq \delta$. Suppose that we removed from $I_0 = [0, \sum a_k]$, n open intervals of lengths $a_1, a_2, \ldots, a_n$. What remains can be written as the union of closed intervals $E_j^{(n)}$, $j = 1, 2, \ldots, n+1$. Since $\sum_{j=1}^{n+1} |E_j^{(n)}| = \sum_{j=n+1}^{\infty} a_j \leq \sum_{j=n+1}^{\infty} r^j \leq \delta$, then $\{E_j^{(n)} : j = 1, 2, \ldots, n+1\}$ is a δ-covering of C_a. Using the Hölder inequality, we have for each $\epsilon > 0$,

$$\begin{aligned} \sum_{j=1}^{n+1} |E_j^{(n)}|^\epsilon &\leq \left(\sum_{j=1}^{n+1} |E_j^{(n)}| \right)^\epsilon (n+1)^{1-\epsilon} \\ &\leq \left(\sum_{j=n+1}^{\infty} r^j \right)^\epsilon (n+1)^{1-\epsilon} = \left(\frac{r^{n+1}}{1-r} \right)^\epsilon (n+1)^{1-\epsilon}. \end{aligned}$$

But then

$$\limsup_{n \to \infty} \sum_{j=1}^{n+1} |E_j^{(n)}|^\epsilon < \infty,$$

which proves that $\dim C_a \leq \epsilon$. Since this is true for every $\epsilon > 0$, we conclude that dim $C_a = 0$. ∎

Using this Proposition, together with Lemma 2.1, we are able to prove the following interesting property.

Proposition 2.3 *Let $a = \{a_n\}$ such that $a_n > 0$ and $\sum a_n < \infty$, then there exists a rearrangement $\sigma(a)$ of a such that dim $C_{\sigma(a)} = 0$.*

Proof. Using Lemma 2.1, we can decompose the sequence a into at most countably many subsequences, all of them having at least geometrical decay. Let $\{\gamma_j\}$ be the family of functions given by Lemma 2.1 and let C_{γ_j} be the Cantor set associated to the subsequence $\{a_{\gamma_j(n)}\}$. Note that, since the sequence $\{a_{\gamma_j(n)}\}$ has at least geometric decay, dim $C_{\gamma_j} = 0$ by Proposition 2.2. Define now

$$t_0 = 0 \qquad \text{and} \qquad t_j = \sum_n a_{\gamma_j(n)}, \quad j = 1, 2, \ldots.$$

Then we have $C_{\gamma_j} \subset [0, t_j]$. Define C to be the union of translates

$$C = \bigcup_j \left(C_{\gamma_j} + (\sum_{k=1}^{j-1} t_k) \right).$$

The set C is a Cantor set and dim $C = 0$ (since it is the at most countable union of Cantor sets of dimension zero.) The lengths of the gaps of C correspond to the terms of the original sequence a. Then, there is a rearrangement $\sigma(a)$ of the sequence, that is associated to the Cantor set C, that is $C = C_{\sigma(a)}$. ∎

3 p-Cantor sets

From this point on, we will again concentrate on the sequence $\lambda = \{k^{-p}\}$. First we will show that, for any rearrangement of this sequence, the $\frac{1}{p}$-Hausdorff measure is finite, immediately providing an upper bound for the Hausdorff dimension.

Proposition 3.1 *Let C_λ be the Cantor set associated to the sequence $\lambda_k = k^{-p}$, $k \in I\!N$, with $p > 1$. Then*

$$H^{\frac{1}{p}}(C_\lambda) \leq \left(\frac{1}{p-1} \right)^{\frac{1}{p}},$$

in particular,

$$\text{dim } C_\lambda \leq \frac{1}{p}.$$

Moreover, if $\sigma(\lambda)$ is any rearrangement of this sequence, then

$$\text{dim } C_{\sigma(\lambda)} \leq \frac{1}{p}.$$

Proof. The proof is analogous to the one of Proposition 2.2. Consider n large enough such that $\sum_{j=n+1}^{\infty} \lambda_j \le \delta$, so that after removing from I_0 open intervals of lengths λ_1, λ_2, ..., λ_n, we have the closed intervals $\{E_j^{(n)}, j = 1, 2, \ldots, n+1\}$, which is a δ-covering of C. Using the Hölder inequality and the Integral Comparison Test for sequences, we have for $0 < s < 1$,

$$\begin{aligned}\sum_{j=1}^{n+1} |E_j^{(n)}|^s &\le \left(\sum_{j=1}^{n+1} |E_j^{(n)}|\right)^s (n+1)^{1-s} = \left(\sum_{j=n+1}^{\infty} j^{-p}\right)^s (n+1)^{1-s} \\ &\le \left(\frac{n^{1-p}}{p-1}\right)^s (n+1)^{1-s} = \left(\frac{1}{p-1}\right)^s \frac{(n+1)^{1-s}}{n^{sp-s}}.\end{aligned}$$

Hence, if $s \ge \frac{1}{p}$,

$$\limsup_{n\to\infty} \sum_{j=1}^{n+1} |E_j^{(n)}|^s \le \left(\frac{1}{p-1}\right)^s < \infty,$$

and therefore, for any $s \ge \frac{1}{p}$, $\mathcal{H}_\delta^s(C_\lambda) < \infty$. In particular, dim $C_\lambda \le \frac{1}{p}$.

To prove the more general case, we can argue as follows. Let $\sigma(\lambda)$ be any rearrangement of $\lambda = \{\lambda_k\}$. Regardless in what order we remove open intervals from I_0, there will be a step m at which the first n intervals of length λ_1, λ_2, ..., λ_n will be removed ($m \ge n$). We consider the $n+1$ closed intervals $E_j^{(n)}$ which are complementary (in I_0) to these removed n-intervals. Clearly $E_j^{(n)}$, $j = 1, \ldots, n+1$ again forms a δ-covering of $C_{\sigma(\lambda)}$ and therefore, the same bounds as before hold. Thus Proposition 3.1 and the upper bounds for Theorems 1.1 and 1.2 are proved. ∎

3.1 The Proof of Theorem 1.1

Since the bounds from above are proved in Proposition 3.1, in order to complete the proof of Theorem 1.1, we need to show that the Cantor set C_λ has positive $\frac{1}{p}$-Hausdorff measure.

The following lemma is the main ingredient in the proof of the theorem. We remind the reader that I_ℓ^k stands for the ℓth interval obtained in the kth step of construction, which was described in the Introduction. The length of the interval I_ℓ^k is given by equation (1), from which we make the observation that, if the gaps form a monotone non-increasing sequence, then so do the diameters of these intervals, that is, for $\ell' \ge \ell$, $|I_\ell^k| \ge |I_{\ell'}^k|$.

Lemma 3.2 *For all* $k = 1, 2, \ldots$ *and* $\ell = 0, 1, \ldots, 2^k - 1$,

$$\lambda_{2^k+\ell+1}\frac{2^p}{2^p-2} \le |I_\ell^k| \le \frac{2^p}{2^p-2}\lambda_{2^k+\ell},$$

and therefore for $\ell' \ge \ell$,

$$1 \le \frac{|I_\ell^k|}{|I_{\ell'}^k|} \le 2^p. \tag{2}$$

Proof. We can rewrite the expression for $|I_\ell^k|$ in (1) as follows

$$|I_\ell^k| = \sum_{h=0}^{\infty}\sum_{j=0}^{2^h-1}\lambda_{2^{k+h}+\ell 2^h+j}.$$

Then

$$\begin{aligned} |I_\ell^k| &\le \sum_{h=0}^{\infty}\frac{2^h}{(2^h(2^k+\ell))^p} = \frac{1}{(2^k+\ell)^p}\sum_{h=0}^{\infty}\left(\frac{1}{2^{p-1}}\right)^h \\ &= \lambda_{2^k+\ell}\frac{2^{p-1}}{2^{p-1}-1} = \lambda_{2^k+\ell}\frac{2^p}{2^p-2}. \end{aligned}$$

The bound from below is obtained in a similar way

$$\begin{aligned} |I_\ell^k| &= \sum_{h=0}^{\infty}\sum_{j=0}^{2^h-1}\lambda_{2^{k+h}+\ell 2^h+j} \\ &\ge \sum_{h=0}^{\infty}\frac{2^h}{2^{hp}\left(2^k+\ell+1-\frac{1}{2^h}\right)^p} \\ &\ge \frac{1}{(2^k+\ell+1)^p}\sum_{h=0}^{\infty}\left(\frac{1}{2^{(p-1)}}\right)^h \\ &= \lambda_{2^k+\ell+1}\frac{2^p}{2^p-2}. \end{aligned}$$

For the inequality (2), since

$$\frac{|I_\ell^k|}{|I_{\ell'}^k|} \le \frac{|I_0^k|}{|I_{2^k-1}^k|},$$

the result follows. ∎

The next lemma is a simple algebraic property of numbers which we will use repeatedly.

Lemma 3.3 *Let a, b, c be arbitrary positive numbers, let $p > 1$ and set $x = a + b + c$. Then*

$$x \leq \frac{2^p}{2^p - 2}\, c \quad \Longrightarrow \quad x^{\frac{1}{p}} \geq a^{\frac{1}{p}} + b^{\frac{1}{p}}.$$

Proof. If $x \leq \frac{2^p}{2^p-2}\, c$, then $\frac{x-c}{2} \leq \frac{x}{2^p}$. Equivalently, $\left(\frac{a+b}{2}\right)^{\frac{1}{p}} \leq \frac{x^{\frac{1}{p}}}{2}$, and, by convexity of the function $x^{\frac{1}{p}}$, $\frac{a^{\frac{1}{p}}+b^{\frac{1}{p}}}{2} \leq \left(\frac{a+b}{2}\right)^{\frac{1}{p}}$, from which the result follows. ∎

Combining the previous lemmas, we obtain the main relation between the intervals of step k and those of step $k+1$.

Lemma 3.4 *For all $k \geq 1$ and $\ell = 0, 1, \ldots, 2^k - 1$,*

$$|I_\ell^k|^{\frac{1}{p}} \geq |I_{2\ell}^{k+1}|^{\frac{1}{p}} + |I_{2\ell+1}^{k+1}|^{\frac{1}{p}}.$$

Proof. By construction and from Lemma 3.2 we have

$$\begin{aligned} |I_\ell^k| &= |I_{2\ell}^{k+1}| + |I_{2\ell+1}^{k+1}| + \lambda_{2^k+\ell} \quad \text{and} \\ |I_\ell^k| &\leq \frac{2^p}{2^p-2}\lambda_{2^k+\ell} \end{aligned}$$

Hence, by Lemma 3.3 the result is obtained. ∎

Lemma 3.5 *Let J be an arbitrary open interval in I_0. Let $k_1 \in I\!N$ be fixed. Then*

$$4|J|^{\frac{1}{p}} \geq \sum_{\ell \,:\, I_\ell^{k_1} \subset J} |I_\ell^{k_1}|^{\frac{1}{p}}.$$

Proof. If $J \cap C_\lambda = \emptyset$, the result is trivial. Now, if $J \cap C_\lambda \neq \emptyset$, and J is open, then there exists some Cantor interval $I_\ell^k \subset J$.

Next observe that if I_ℓ^k and $I_{\ell+1}^k$ are consecutive intervals from step k and obtained from one interval in step $k-1$, (we shall say that these two intervals have a common "father"), then ℓ is even. If we consider two consecutive intervals of step k not having a common father, i.e. I_ℓ^k and $I_{\ell+1}^k$ with ℓ odd, let I be the minimal closed interval containing I_ℓ^k and $I_{\ell+1}^k$. Then $I - \left(I_\ell^k \cup I_{\ell+1}^k\right)$ is a gap of a previous step, i.e.

$$|I| = |I_\ell^k| + \lambda_{2^s+r} + |I_{\ell+1}^k|$$

with $s \leq k-2$ and $r = [\frac{\ell}{2^{k-s}}] < \frac{\ell}{2^{k-s}}$. We want to prove that $|I|^{\frac{1}{p}} \geq |I_\ell^k|^{\frac{1}{p}} + |I_{\ell+1}^k|^{\frac{1}{p}}$. By Lemma 3.2 we have

$$|I_\ell^k| \leq \frac{2^p}{2^p-2}\lambda_{2^k+\ell},$$

and since

$$\lambda_{2^k+\ell} = \frac{1}{(2^k+\ell)^p} \le \frac{1}{(2^{k-s})^p\,(2^s+r)^p},$$

we conclude that

$$|I_\ell^k| \le \frac{1}{(2^{k-s})^p}\frac{2^p}{2^p-2}\lambda_{2^s+r}.$$

Since $|I_{\ell+1}^k| < |I_\ell^k|$ and $k-s \ge 2$, we have

$$|I| \le \left(\frac{2}{(2^{k-s})^p}\frac{2^p}{2^p-2}+1\right)\lambda_{2^s+r} \le \frac{2^p}{2^p-2}\lambda_{2^s+r}.$$

Thus we can apply Lemma 3.3 to obtain

$$|I|^{\frac{1}{p}} \ge |I_\ell^k|^{\frac{1}{p}} + |I_{\ell+1}^k|^{\frac{1}{p}}.$$

Define now $k_0 := \min\{k \in \mathbb{N} : I_\ell^k \subset J \text{ for some } 0 \le \ell < 2^k-1\}$. First we observe that the interval J can contain at most two intervals of step k_0. We will prove the case in which J contains exactly two, the other case can be proved similarly. (Note that by definition of k_0, J must contain at least one interval of step k_0.)

Let $I_\ell^{k_0}$ and $I_{\ell+1}^{k_0}$ be the intervals in J. Then ℓ is odd and only four Cantor intervals of step k_0 can intersect J. These are $I_{\ell-1}^{k_0}$, $I_\ell^{k_0}$, $I_{\ell+1}^{k_0}$, and $I_{\ell+2}^{k_0}$. Let $\tilde{I}$ be the smallest interval containing $I_\ell^{k_0}$ and $I_{\ell+1}^{k_0}$. Using (3.1) and since $J \supset \tilde{I}$, we have

$$|J|^{\frac{1}{p}} \ge |\tilde{I}|^{\frac{1}{p}} \ge |I_\ell^{k_0}|^{\frac{1}{p}} + |I_{\ell+1}^{k_0}|^{\frac{1}{p}}. \tag{3}$$

Now using Lemma 3.2 we know that

$$2^p|I_\ell^{k_0}| \ge |I_{\ell-1}^{k_0}|,$$

that is,

$$2|J|^{\frac{1}{p}} \ge 2|I_\ell^{k_0}|^{\frac{1}{p}} \ge |I_{\ell-1}^{k_0}|^{\frac{1}{p}}. \tag{4}$$

Finally, since $|I_{\ell+1}^{k_0}| \ge |I_{\ell+2}^{k_0}|$ and $I_{\ell+1}^{k_0} \subset J$,

$$|J|^{\frac{1}{p}} \ge |I_{\ell+1}^{k_0}|^{\frac{1}{p}} \ge |I_{\ell+2}^{k_0}|^{\frac{1}{p}}. \tag{5}$$

From (3), (4), and (5) we get

$$4|J|^{\frac{1}{p}} \ge |I_{\ell-1}^{k_0}|^{\frac{1}{p}} + |I_\ell^{k_0}|^{\frac{1}{p}} + |I_{\ell+1}^{k_0}|^{\frac{1}{p}} + |I_{\ell+2}^{k_0}|^{\frac{1}{p}}.$$

Now, if $k_1 \ge k_0$, using Lemma 3.4 inductively, we have

$$4|J|^{\frac{1}{p}} \ge \sum_{\ell : I_\ell^{k_1} \subset J} |I_\ell^{k_1}|^{\frac{1}{p}}$$

and if $k_1 < k_0$ there are no intervals $I_\ell^{k_1} \subset J$, $\ell = 0, \ldots, 2^{k_1} - 1$ and the inequality is obvious. This completes the proof. ∎

We are now ready to prove Theorem 1.1. Let $F = \{F_i\}_{i\in\mathbb{N}}$ be a covering of C_λ with open intervals of length less than δ,

$$\bigcup F_i \supset C_\lambda \quad \text{and} \quad \operatorname{diam}(F_i) < \delta,\ \forall\ i.$$

Since C_λ is compact, let $\{F_{h_j} = (\alpha_j, \beta_j)\}_{j=1}^m$ be a finite subcovering of C_λ, $F_{h_j} \in F$, $j = 1, \ldots, m$.

Let $\varepsilon > 0$. Since $\mathbb{R} \setminus C_\lambda$ is dense in $\mathbb{R}$, we can construct open intervals, $E_j = (a_j, b_j)$, $j = 1, \ldots, m$ such that

$$F_{h_j} \subset E_j, \quad |E_j|^{\frac{1}{p}} < |F_{h_j}|^{\frac{1}{p}} + \frac{\varepsilon}{m} \quad \text{and} \quad a_j, b_j \notin C_\lambda.$$

Therefore,

$$\sum_{j=1}^{m} |E_j|^{\frac{1}{p}} < \sum_{j=1}^{m} |F_{h_j}|^{\frac{1}{p}} + \varepsilon.$$

Since ε is arbitrary, if

$$\sum_{i=1}^{m} |E_i|^{\frac{1}{p}} \geq \frac{1}{4} \frac{1}{(2^p - 2)^{\frac{1}{p}}}, \tag{6}$$

we obtain the desired lower bound for the Hausdorff measure. But since $a_i, b_i \notin C_\lambda$, for k large enough, we can make $|I_\ell^k|$ so small that, for all $0 \leq \ell < 2^k$, $I_\ell^k \subset E_i$ for some i.

Therefore, using Lemma 3.5,

$$\sum_{i=1}^{m} |E_i|^{\frac{1}{p}} \geq \sum_{i=1}^{m} \frac{1}{4} \left(\sum_{\ell\,:\, I_\ell^k \subset E_i} |I_\ell^k|^{\frac{1}{p}} \right) \geq \frac{1}{4} \sum_{\ell=0}^{2^k-1} |I_\ell^k|^{\frac{1}{p}}.$$

Since $|I_\ell^k| \geq |I_{\ell+1}^k|$, we get

$$\frac{1}{4} \sum_{\ell=0}^{2^k-1} |I_\ell^k|^{\frac{1}{p}} \geq \frac{2^k}{4} |I_{2^k-1}^k|^{\frac{1}{p}},$$

and, using the estimate of Lemma 3.2, we get (6). ∎

3.2 The Cantor sets $C_{\tilde{\lambda}}$

In this section we are going to prove the following Proposition.

Proposition 3.6 *Let* $\lambda = \{\lambda_k = k^{-p}\}$. *For each* $0 \leq s \leq \frac{1}{p}$ *there exists a rearrangement* $\sigma_s(\lambda)$ *of* λ *such that*

$$\mathcal{H}^s(C_{\sigma_s(\lambda)}) > 0.$$

To prove this, we first find the Hausdorff dimension of the Cantor set $C_{\tilde{\lambda}}$ where $\tilde{\lambda}$ is a particular subsequence of the original p-sequence. This result together with the following lemma will complete the proof.

Lemma 3.7 *If* $\tilde{\lambda}$ *is a subsequence of* $\lambda = \{\lambda_k = k^{-p}\}$ *such that dim* $C_{\tilde{\lambda}} = s$, *then there exists a rearrangement* $\sigma(\lambda) = \sigma_s(\lambda)$ *of* λ, *such that*

$$dim\ C_{\sigma_s(\lambda)} = dim\ C_{\tilde{\lambda}} = s.$$

Proof. Let $\gamma = \{\gamma_k\}$ be the subsequence obtained from λ after deleting the terms of the subsequence $\tilde{\lambda}$. By Proposition 2.3 there exists a rearrangement $\sigma(\gamma)$ of γ, such that dim $C_{\sigma(\gamma)} = 0$. Let now $t_1 = \sum_k \gamma_k$, then we have $C_{\sigma(\gamma)} \subset [0, t_1]$. Define

$$C = C_{\sigma(\gamma)} \cup (C_{\tilde{\lambda}} + \{t_1\}).$$

The set C is a Cantor set and dim $C = s$ (since it is the union of one Cantor set of dimension s and one of dimension 0). The lengths of the gaps of C correspond to the terms of the original sequence λ. Then, there is a rearrangement σ of the sequence, that is associated to the Cantor set C, that is $C = C_{\sigma(\lambda)}$. ∎

It is now clear that, in order to obtain Proposition 3.6, it suffices for each s to find a particular subsequence $\tilde{\lambda}$, such that dim $C_{\tilde{\lambda}} = s$.

Let $x \geq 2$ be a fixed real number. We define a subsequence $\tilde{\lambda}_m$ by the following relation: using the fact that m can be decomposed uniquely into $m = 2^k + j$ with $k \geq 0$ and $j = 0, 1, \ldots, 2^{k-1}$, and using the notation $[x]$ to denote the greatest integer in x, we set

$$\tilde{\lambda}_m = \tilde{\lambda}_{2^k+j} = \lambda_{[x^k]+j} = \left(\frac{1}{[x^k]+j}\right)^p.$$

Since this subsequence is completely determined by x, and in order to avoid cumbersome notation, we denote by C_x the Cantor set $C_{\tilde{\lambda}}$.

Theorem 3.8 *With notation as above and with* $\alpha(p,x) := \frac{\log 2}{p \log x}$, *we have*

$$c\left(\frac{x^p}{x^p-2}\right)^{\alpha} \leq \mathcal{H}^{\alpha}(C_x) \leq \left(\frac{4^p}{2^p-2}\right)^{\alpha},$$

for some positive constant c *depending only on* x *and* p. *Hence,* C_x *is an* α*-set.*

Proof. The proof of this theorem has the same flavor as that of Theorem 1.1. However, since we are dealing with subsequences of the original p-series, the estimates need some more careful consideration. Again we will split the proof of the theorem into two separate statements – one for the upper bound and one for the lower bound. We use the following notation: all quantities introduced in the proofs of Theorem 1.1 and Proposition 3.6 corresponding to the sequence $\{\lambda_j\}$ will be used with the sign ˜ to denote quantities corresponding to $\{\tilde{\lambda}_j\}$.

Proposition 3.9 *Let* C_x *be the Cantor set associated to the sequence* $\tilde{\lambda} = \{\tilde{\lambda}_{2^k+j} = \lambda_{[x^k]+j},\ k \in I\!N, j = 0, 1, \ldots, 2^k - 1\}$. *Then*

$$H^{\alpha}(C_x) \leq \left(\frac{4^p}{2^p-2}\right)^{\alpha},$$

hence dim $C_x \leq \alpha$.

Proof. The proof goes along the lines of the proof of Proposition 3.1, only instead of λ_j, we now have $\tilde{\lambda}_j$. Suppose that $n = 2^k - 1$ is sufficiently large such that $\sum_{j=n+1}^{\infty} \tilde{\lambda}_j \leq \delta$. The remaining intervals in $\tilde{I}_0 = [0, \sum_j \tilde{\lambda}_j]$, after removing the open intervals of lengths $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, ..., $\tilde{\lambda}_n$, are contained in the union of intervals of the collection $\{\tilde{E}_j^{(n)} : j = 1, 2, \ldots, n+1\}$ which is therefore a δ-covering of C_x. Using again the Hölder inequality, we have

$$\sum_{j=1}^{n+1} |\tilde{E}_j^{(n)}|^s \leq \left(\sum_{j=1}^{n+1} |\tilde{E}_j^{(n)}|\right)^s (n+1)^{1-s} = \left(\sum_{j=n+1}^{\infty} \tilde{\lambda}_j\right)^s (n+1)^{1-s}.$$

We now estimate the quantity

$$\sum_{j=n+1}^{\infty} \tilde{\lambda}_j = \sum_{m=k}^{\infty} \sum_{j=0}^{2^m-1} \tilde{\lambda}_{2^m+j}.$$

We have

$$\sum_{m=k}^{\infty} \sum_{j=0}^{2^m-1} \tilde{\lambda}_{2^m+j} \quad = \quad \sum_{m=k}^{\infty} \sum_{j=0}^{2^m-1} \lambda_{[x^m]+j} = \sum_{m=k}^{\infty} \sum_{j=0}^{2^m-1} \frac{1}{([x^m]+j)^p}$$

$$\begin{aligned}
&\leq \sum_{m=k}^{\infty} \frac{2^m}{[x^m]^p} \leq \sum_{m=k}^{\infty} \left(\frac{1}{x^m - 1}\right)^p 2^m \\
&\leq \sum_{m=k}^{\infty} 2^p \left(\frac{2}{x^p}\right)^m = \left(\frac{2}{x^p}\right)^k 2^p \frac{x^p}{x^p - 2} \\
&\leq K(p) \left(\frac{2}{x^p}\right)^k,
\end{aligned}$$

where $K(p) = 4^p/(2^p - 2)$ is a bound for $(2x)^p/(x^p - 2)$. Since $2^k = n+1$ and $\alpha = \frac{\log 2}{p \log x}$, we have that $x^{pk} = (n+1)^{1/\alpha}$, and

$$\sum_{j=1}^{n+1} |\tilde{E}_j^{(n)}|^s \leq K^s(p) \frac{(n+1)^s}{x^{pks}} (n+1)^{1-s} = K^s(p)(n+1)^{1-s/\alpha},$$

which again, if $s \geq \alpha$, yield,

$$\limsup_{n \to \infty} \sum_{j=1}^{n+1} |\tilde{E}_j^{(n)}|^s < \left(\frac{4^p}{2^p - 2}\right)^s < +\infty.$$

In particular,

$$\dim C_x \leq \alpha. \qquad \blacksquare$$

To complete the proof that our Cantor set is an α-set, we must show that the Hausdorff measure of C_x, $\mathcal{H}^\alpha(C_x)$, is positive. The idea is to repeat the proofs used for the set C_λ. However, it is immediately seen, that the same estimate will not work in this case, and in fact, this estimate is much harder to obtain.

Proposition 3.10 *Let $p > 1$, $x > 2$ and let $\tilde{\lambda} = \{\tilde{\lambda}_k\}_{k \in \mathbb{N}}$ be the sequence defined by $\tilde{\lambda}_{2^k + j} = \left(\frac{1}{[x^k] + j}\right)^p$. If C_x is the Cantor set associated to the sequence $\tilde{\lambda}$, and $\alpha = \frac{\log 2}{p \log x}$, then there exists a positive constant c, which depends only on x and p, such that*

$$\mathcal{H}^\alpha(C_x) \geq c \left(\frac{x^p}{x^p - 2}\right)^\alpha.$$

This Proposition is the analogue of Proposition 3.6, which was proved by combining Lemmas 3.2, 3.3, 3.4 and 3.5. We will need to provide the analogues of these lemmas for this new sequence. Note that Lemma 3.3 is independent of the sequence.

Since

$$\tilde{I}_\ell^k = \sum_{h=0}^{\infty} \sum_{j=0}^{2^h-1} \tilde{\lambda}_{2^{k+h}+\ell 2^h+j},$$

using the same arguments as in Lemma 3.2 we immediately have:

Lemma 3.11 *For every fixed $x > 2$ and all $k \geq 1$ and $\ell = 0, \ldots, 2^k - 1$,*

$$\frac{x^p}{x^p - 2} \cdot \frac{1}{x^{(k+1)p}} \leq |\tilde{I}_\ell^k| \leq \frac{x^p}{x^p - 2} \left(\frac{1}{x^k - 1} \right)^p$$

and therefore, for $\ell' \geq \ell$,

$$1 \leq \frac{|\tilde{I}_\ell^k|}{|\tilde{I}_{\ell'}^k|} \leq (2x)^p.$$

Since Lemma 3.11 provides an estimate which is not as precise as that of Lemma 3.2, we have now a weaker version of Lemma 3.4. This is probably the essence of the lower bound estimate for the Hausdorff measure. Once we have a relation between the sizes of the intervals of one step with those of the next step, we are able to proceed with the bounds for the Hausdorff measure.

Lemma 3.12 *For all $k \in \mathbb{N}$ and all $\ell = 0, 1, \ldots, 2^k - 1$*

$$|\tilde{I}_\ell^k|^\alpha \geq B_k^{-\alpha} \left(|\tilde{I}_{2\ell}^{k+1}|^\alpha + |\tilde{I}_{2\ell+1}^{k+1}|^\alpha \right)$$

where the sequence B_k satisfies, $B_k > 1$ and $\prod_{k=1}^{\infty} B_k^{-\alpha} = \zeta$, with $\zeta = \zeta(p, x) > 0$.

Proof. From Lemma 3.11 we get the following estimate

$$|\tilde{I}_\ell^k| \leq \frac{x^p}{x^p - 2} \left(\frac{[x^k] + \ell}{x^k - 1} \right)^p \tilde{\lambda}_{2^k+\ell},$$

and since $\ell \leq 2^k - 1$,

$$|\tilde{I}_\ell^k| \leq \frac{x^p}{x^p - 2} \left(\frac{x^k + 2^k - 1}{x^k - 1} \right)^p \tilde{\lambda}_{2^k+\ell}.$$

Since $|\tilde{I}_\ell^k| = |\tilde{I}_{2\ell}^{k+1}| + \tilde{\lambda}_{2^k+\ell} + |\tilde{I}_{2\ell+1}^{k+1}|$, this gives the estimate

$$|\tilde{I}_\ell^k| \left(1 - \frac{x^p - 2}{x^p} \left(\frac{x^k - 1}{x^k + 2^k - 1} \right)^p \right) \geq |\tilde{I}_{2\ell}^{k+1}| + |\tilde{I}_{2\ell+1}^{k+1}|. \tag{7}$$

Now put

$$B_k = 1 + \left(\frac{x^p}{2} - 1\right)\left(1 - \left(\frac{x^k - 1}{x^k + 2^k - 1}\right)^p\right).$$

By simple algebra

$$\frac{2}{x^p} B_k = 1 - \frac{x^p - 2}{x^p}\left(\frac{x^k - 1}{x^k + 2^k - 1}\right)^p.$$

Since $x^{p\alpha} = 2$, raising both sides of (7) to the power α, and applying convexity arguments as in Lemma 3.3, we get

$$\frac{|\tilde{I}_\ell^k|^\alpha}{2} B_k^\alpha \geq \frac{\left(|\tilde{I}_{2\ell}^{k+1}|^\alpha + |\tilde{I}_{2\ell+1}^{k+1}|^\alpha\right)}{2}.$$

Therefore

$$|\tilde{I}_\ell^k|^\alpha \geq B_k^{-\alpha}\left(|\tilde{I}_{2\ell}^{k+1}|^\alpha + |\tilde{I}_{2\ell+1}^{k+1}|^\alpha\right).$$

Since $B_k > 1$, to see that $\prod_{k=1}^\infty B_k^{-\alpha} > 0$ it is enough to see that $\prod_{k=1}^\infty B_k < +\infty$. But $x > 2$ and $p > 1$ and so we can write

$$\begin{aligned} 1 - \left(\frac{x^k - 1}{x^k + 2^k - 1}\right)^p &= 1 - \frac{1}{\left(1 + \frac{2^k}{x^k - 1}\right)^p} \leq 1 - \frac{1}{\left(1 + 2(\frac{2}{x})^k\right)^p} \\ &= \frac{\left(1 + 2(\frac{2}{x})^k\right)^p - 1^p}{\left(1 + 2(\frac{2}{x})^k\right)^p} \leq \frac{p\left(1 + 2(\frac{2}{x})^k\right)^{p-1} 2(\frac{2}{x})^k}{\left(1 + 2(\frac{2}{x})^k\right)^p} \leq 2p(\frac{2}{x})^k. \end{aligned}$$

The second inequality follows by an application of the Mean Value Theorem. Using this, we have that the following series converges

$$\sum_{k=1}^\infty \left(\frac{x^p}{2} - 1\right)\left(1 - \left(\frac{x^k - 1}{x^k + 2^k - 1}\right)^p\right) \leq 2p\left(\frac{x^p}{2} - 1\right)\sum_k^\infty \left(\frac{2}{x}\right)^k < +\infty,$$

and therefore $\prod_{k=1}^\infty B_k < +\infty$. ∎

Note that the preceding proof fails for the case $x = 2$ because the estimate in Lemma 3.11 is too imprecise. We now prove the analogue of Lemma 3.5.

Lemma 3.13 *Let $\tilde{J}$ be an arbitrary open interval in $\tilde{I}_0$. Let $k \in \mathbb{N}$ be fixed and again let $\alpha = \log 2/(p \log x)$. Then there exists c, independent of k, such that*

$$c|\tilde{J}|^\alpha \geq \sum_{\ell : \tilde{I}_\ell^k \subset \tilde{J}} |\tilde{I}_\ell^k|^\alpha.$$

Proof. If $\tilde{J} \cap C_{\tilde{\lambda}} = \emptyset$, the result is trivial. Otherwise, define $k_0 := \min\{k \in \mathbb{N} : \tilde{J}$ contains an interval of step $k\}$. As before, $\tilde{J}$ can contain at most two intervals of step k_0. Again we will only prove the case in which $\tilde{J}$ contains exactly two.

Let $\tilde{I}_\ell^{k_0}$ and $\tilde{I}_{\ell+1}^{k_0}$ be the intervals in $\tilde{J}$. Then ℓ is odd and only four Cantor intervals of step k_0 can intersect $\tilde{J}$. These are $\tilde{I}_{\ell-1}^{k_0}$, $\tilde{I}_\ell^{k_0}$, $\tilde{I}_{\ell+1}^{k_0}$, and $\tilde{I}_{\ell+2}^{k_0}$. Since $\tilde{J} \supset \tilde{I}_\ell^{k_0}$ and $|\tilde{I}_\ell^{k_0}| \geq |\tilde{I}_{\ell+1}^{k_0}| \geq |\tilde{I}_{\ell+2}^{k_0}|$, we have

$$3\,|\tilde{J}|^\alpha \geq |\tilde{I}_\ell^{k_0}|^\alpha + |\tilde{I}_{\ell+1}^{k_0}|^\alpha + |\tilde{I}_{\ell+2}^{k_0}|^\alpha$$

Now using Lemma 3.11, $(2x)^p |\tilde{I}_\ell^{k_0}| \geq |\tilde{I}_{\ell-1}^{k_0}|$, that is

$$(2x)^{p\alpha} |\tilde{J}|^\alpha \geq (2x)^{p\alpha} |\tilde{I}_\ell^{k_0}|^\alpha \geq |\tilde{I}_{\ell-1}^{k_0}|^\alpha.$$

Hence

$$(3 + (2x)^{p\alpha})|\tilde{J}|^\alpha \geq |\tilde{I}_{\ell-1}^{k_0}|^\alpha + |\tilde{I}_\ell^{k_0}|^\alpha + |\tilde{I}_{\ell+1}^{k_0}|^\alpha + |\tilde{I}_{\ell+2}^{k_0}|^\alpha.$$

Now inductively we apply Lemma 3.12 and get,

$$(3 + (2x)^{p\alpha})|\tilde{J}|^\alpha \geq (B_{k_0} B_{k_0+1} \dots B_{k-1})^{-\alpha} \sum_{\ell : \tilde{I}_\ell^k \subset \tilde{J}} |\tilde{I}_\ell^k|^\alpha.$$

Since

$$(B_{k_0} B_{k_0+1} \dots B_{k-1})^{-\alpha} \geq \prod_{k=1}^{\infty} B_k^{-\alpha} \equiv \zeta > 0,$$

we have

$$\frac{3 + (2x)^{p\alpha}}{\zeta} |\tilde{J}|^\alpha \geq \sum_{\ell : \tilde{I}_\ell^k \subset \tilde{J}} |\tilde{I}_\ell^k|^\alpha.$$

∎

We are now ready to prove Propostion 3.10.

Proof (of Proposition 3.10 and hence Theorem 1.2). As in the proof of Theorem 1.1, we choose a finite δ-covering consisting of intervals as close as we wish to an arbitrary covering of C_x. We must bound the Hausdorff α measure of this covering.

Let $F = \{F_i\}_{i \in \mathbb{N}}$ be a covering of C_x with open intervals of length less than δ,

$$\bigcup F_i \supset C_x \quad \text{and} \quad \operatorname{diam}(F_i) < \delta,\ \forall\, i.$$

Again, let $\{F_{h_j} = (\alpha_j, \beta_j)\}_{j=1}^m$ be a finite subcovering of C_x, $F_{h_j} \in F, j = 1, \dots, m$ and for $\varepsilon > 0$ choose open intervals, $\tilde{E}_j = (a_j, b_j), j = 1, \dots, m$ such that

$$F_{h_j} \subset \tilde{E}_j, \quad |\tilde{E}_j|^{\frac{1}{p}} < |F_{h_j}|^{\frac{1}{p}} + \frac{\varepsilon}{m} \quad \text{and} \quad a_j, b_j \notin C_x.$$

Therefore,

$$\sum_{j=1}^{m} |\tilde{E}_j|^\alpha < \sum_{j=1}^{m} |F_{h_j}|^\alpha + \varepsilon.$$

Hence, the proof is completed by showing that $\sum_{i=1}^{m} |\tilde{E}_i|^\alpha \geq c\left(\frac{x^p}{x^p-2}\right)^\alpha$ for some constant c. But again, for k large enough, $|\tilde{I}_\ell^k|$ is so small that, for all ℓ, $\tilde{I}_\ell^k \subset \tilde{E}_i$ for some i. Therefore using Lemma 3.13, and defining $2c = \zeta/(3+(2x)^{p\alpha})$, we have

$$\sum_{i=1}^{m} |\tilde{E}_i|^\alpha \geq \sum_{i=1}^{m} \frac{\zeta}{3+(2x)^{p\alpha}} \left(\sum_{\ell\,:\,\tilde{I}_\ell^k \subset \tilde{E}_i} |\tilde{I}_\ell^k|^\alpha \right) \geq 2c \sum_{\ell=0}^{2^k-1} |\tilde{I}_\ell^k|^\alpha.$$

Since $|\tilde{I}_\ell^k| \geq |\tilde{I}_{\ell+1}^k|$, we get

$$\sum_{\ell=0}^{2^k-1} |\tilde{I}_\ell^k|^\alpha \geq 2^k |\tilde{I}_{2^k-1}^k|^\alpha,$$

and using the estimate of Lemma 3.11 and the fact that $x^{p\alpha} = 2$, we get

$$\sum_{i=1}^{m} |\tilde{E}_i|^\alpha \geq c 2^{k+1} \left(\frac{x^p}{x^p-2}\right)^\alpha \frac{1}{x^{(k+1)p\alpha}} = c\left(\frac{x^p}{x^p-2}\right)^\alpha. \qquad \blacksquare$$

4 A generalization

In this section we generalize to the case in which $r-1$ intervals are removed at each step, $r \geq 2$; the case $r = 2$ was considered above. Thus from I_0 $r-1$ open intervals are removed leaving the r closed intervals $I_0^1, \ldots, I_{r-1}^1$. From each of these, $r-1$ open intervals are removed and so at end of the second step of the construction there remains the r^2 closed intervals $I_0^2, \ldots, I_{r^2-1}^2$. Now continue the construction in this fashion. We will denote the associated Cantor set by C_r.

Surprisingly, the results of the previous sections remain true. Since the proofs can be obtained by the same methodology as presented in the previous sections, we will only state one of the generalized results in this direction.

Note that when carrying out the proofs, the role played by 2^k is now played by r^k since before we had 2^k intervals at a given step k, and now we have r^k intervals at that same step.

Theorem 4.1 *Let* $\lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ *be defined by* $\lambda_k = \left(\frac{1}{k}\right)^p$, $p > 1$. *Then for* $r \geq 2$,

$$dim\ C_r = \frac{1}{p}.$$

Moreover

$$0 < \mathcal{H}^{\frac{1}{p}}(C_r) < +\infty.$$

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