Return On Investment I – Averaging

Return On Investment I – Averaging

Outline:

- Definition of rate of return.
- Compounding – nominal vs effective rate.
- Continuous compounding and $e = 2.71828\ldots$.
- Geometric Average.
- Arithmetic Average.

Definition of Rate of Return

We use the terms "Principle", "interest", and "interest rate" but it applies just as well to "amount invested", "return", and "rate of return".

Let P be amount invested and let ΔP be the increase in value, the return. Of course $\Delta P < 0$ if there is a loss. And let the investment be in force over the time period t in years. Then the rate of return is the return per dollar per year,

$$r = \frac{\Delta P}{P * t}.$$

E.g. a return of \$8 on \$100 invested over 1 year is 8%.

Multiple Periods

Turning the equation around, the amount A_1 one has after $t = 1$ year is

$$A_1 = P + \Delta P = P + Pr = P(1 + r).$$

After $t = 2$ years is (interest on interest)

$$A_2 = (P(1 + r))\,(1 + r) = P(1 + r)^2.$$

After t years (could be fractional, e.g. 3.5)

$$A_t = P(1+r)^t.$$

Finding r

So an investment of P which becomes the amount $A = P + \Delta P$ (equaling the investment plus the return) over any time t (2/3 of a year, 4 and 1/2 years, whatever) has the rate of return given by

$$r = \left(\frac{A}{P}\right)^{1/t} - 1 = \left(1 + \frac{\Delta P}{P}\right)^{1/t} - 1.$$

Compounding

Problem: \$100 invested at 8% compounded quarterly.

Use our basic equation: $A = P(1 + r)^t$, now t counts quarters, i.e. 3 month periods, and r is the rate per quarter, $r = 8/4 = 2\%$. Over $t = 4$ quarters this is

$$A = 100 * (1 + .02)^4 = 108.24.$$

So the interest on interest earned \$0.24 and actual rate of return is 8.24%.

We say $r_N = 8\%$ is the "nominal" rate of return while $r_E = 8.24\%$ is the "effective" rate of return.

What if we compound daily, 365 times per year; over one year we have

$$A = 100 * (1 + \frac{0.08}{365})^{365} = 108.3277\ldots$$

or \$108.33. The effective rate is 8.33%.

Continuous Compounding

Let r be the nominal rate, P the investment and suppose it is compounded k times per year with $k \longrightarrow \infty$, so over 1 year

$$A_1 = \lim_{k \to \infty} P(1 + \frac{r}{k})^k = Pe^r,$$

the exponential where $e = 2.71828\ldots$. (This formula was discovered in ancient times.)

E.g. \$100 at 8% nominal over 1 year equals 8.33%

effective rate

$$100 * e^{0.08} = 108.3287\ldots.$$

Rate of Return Here

Over arbitrary time t (not just one year)

$$A_t = Pe^{rt}.$$

So the nominal rate is

$$r = \frac{1}{t} \log \frac{A_t}{P} = \frac{1}{t} \log \left(1 + \frac{\Delta P}{P} \right).$$

where $\log$ is the natural logarithm function.

Average Return, Discrete Compounding

Suppose the investment covers several quarters each with a different rate of return, e.g. r_1, r_2, ..., r_n. Then the amount after these n quarters is

$$A = P(1 + r_1)(1 + r_2)\dots(1 + r_n).$$

We want the average quarterly rate $\bar{r}$ giving the same return. So $A = P(1 + \bar{r})^n$, solve for $\bar{r}$

$$\bar{r} = ((1 + r_1)(1 + r_2)\dots(1 + r_n))^{1/n} - 1.$$

This is called the *geometric average*.

Average Return, unequal periods

Maybe the last period is only partial, then what? More generally let quartely rate r_1 apply for time t_1 (measured in quarters), r_2 apply for time t_2 and so on. Then, putting $T = t_1 + t_2 + \ldots + t_n$,

$$\bar{r} = (1 + r_1)^{\frac{t_1}{T}}(1 + r_2)^{\frac{t_2}{T}} \ldots (1 + r_n)^{\frac{t_n}{T}} - 1.$$

Again the geometric average.

Average Return, Continuous Compounding

Again let the investment cover several time periods t_1, t_2, ..., t_n (measured in years) with various (nominal) rates of return r_1, r_2, ..., r_n over these time periods. Then the amount we have at the end is

$$\begin{aligned} A &= Pe^{r_1 t_1} e^{r_2 t_2} \ldots e^{r_n t_n} \\ &= Pe^{r_1 t_1 + r_2 t_2 + \ldots + r_n t_n}. \end{aligned}$$

Again with $T = t_1 + t_2 + \ldots + t_n$, the average rate of return is

$$\bar{r} = \frac{1}{T}\left(r_1 t_1 + r_2 t_2 + \ldots + r_n t_n\right).$$

Note this is the ordinary *arithmetic* average.

As you can see, continuous compounding is easier to work with.