Kelly's Problem

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Outline:

- Some Basic, Conceptual Math Facts.
- The Solution for even payoffs.
- The Solution for uneven payoffs.

Bernoulli Trials

Bernoulli Trial: a two outcome probability experiment; the probability of success is p, the probability of failure is $1 - p$.

If the amt gained is $G > 0$ and lost is $L < 0$ then the *expected* payoff is $E = Gp + L(1 - p)$.

Function maximization

If y is a function of x, $y = f(x)$, then the maximum value of y occurs where the tangent line is horizontal, this is where the derivative of f is 0,

$$f'(x) = 0.$$

There are various other technical facts I leave to you to look up.

Basic Kelly's Problem

Our fortune is X, we bet fraction f, we win the bet w/prob p and lose w/prob $1-p$; we know that $p > 1/2$.

Initial fortune, X_0, after one play

$$X_1 = \begin{cases} (1-f)X_0 & \text{if lose} \\ (1+f)X_0 & \text{if win} \end{cases}$$

$$X_1 = B_1 X_0, \quad B_1 = \begin{cases} 1-f & \text{if lose} \\ 1+f & \text{if win} \end{cases}.$$

The Basic Problem

$$X_N = B_N B_{N-1} \dots B_1 X_0$$

Key Step: maximize $W_N = \log(X_N)$

$$W_N = \sum_{i=1}^{N} \log B_i + \log X_0.$$

Let

$$Y_i = \log B_i = \begin{cases} \log(1-f) & \text{w/prob } 1-p \\ \log(1+f) & \text{w/prob } p \end{cases}$$

then

$$E(Y_i) = (1-p)\log(1-f) + p\log(1+f)$$

and

$$E(W_N) = N\left[(1-p)\log(1-f) + p\log(1+f)\right] + \log X_0.$$

The Basic Problem

Maximum expectation when the derivative is zero,

$$0 = \frac{-(1-p)}{(1-f)} + \frac{p}{(1+f)} = \frac{-(1-p)(1+f) + p(1-f)}{1-f^2}.$$

Gives

$$f = 2p - 1.$$

With this for f the expected growth is

$$E(W_N) = N\left(\log 2 + (1-p)\log(1-p) + p\log(p)\right) + \log X_0.$$

Example.

If the probability of winning is $p = 0.6$, then the betting fraction is $f = 2(.6) - 1 = .2$ or 20%. The log grow rate is $E(W_N) = 0.02N + \log X_0$ so our fortune grows like

$$e^{.02N} = 1.02^N.$$

Alternative derivation

The average rate of return $\bar{r}$ satisfies $(1 + \bar{r})^N = X_N$. Maximize this; equivalently, maximize $1 + \bar{r}$; equivalently maximize $\log(1 + \bar{r})$. It leads to the same equation.

Payoffs uneven, information partial

Daniel Madeja, 23 April 2007, "Marble Game"

Now we allow the investment to win or lose multiple times the bet. Suppose we only know:

- We can lose up to 5 times our bet
- the expected payoff is 0.9.

Thus we cannot bet more than 1/5 our fortune and we should not bet even that since play continues indefinitely.

Payoffs uneven, information partial

"Worst case" to match the expected payoff

$$0.9 = -5(1-p) + 1p, \qquad \text{gives } p = 59/60$$

Note the rule is no longer $f = 2p - 1$ since that is for an even payoff. We must re-derive.

Payoffs uneven, information partial

Now $X_{i+1} = (1 - 5f + 6fB_i)X_i$ with $B_i = 0$ w/prob $1/60$ and $B_i = 1$ w/prob $59/60$.

$$W_N = \log X_N = \sum_1^N Y_i + \log X_0$$

where

$$Y_i = \begin{cases} \log(1-5f) & \text{w/prob } 1-p \\ \log(1+f) & \text{w/prob } p \end{cases}$$

Payoffs uneven, information partial

Solve for maximum W_N,

$$0 = \frac{-5(1-p)}{1-5f} + \frac{p}{1+f}$$

gives

$$f = \frac{6}{5}p - 1, \quad \text{for } p = \tfrac{59}{60},\ f = .18$$

Payoffs uneven, complete information

payoff	-5x	-2x	-1x	+1x	+2x	+10x
prob.	1/10	1/10	2/10	2/10	3/10	1/10

$$E(\text{payoff}) = -5\frac{1}{10} - 2\frac{1}{10} - 1\frac{2}{10} + 1\frac{2}{10} + 2\frac{3}{10} + 10\frac{1}{10}$$
$$= 0.9.$$

Payoffs uneven, complete information

Proceed as before

$$W_N = \log X_N = \sum_{1}^{N} Y_i + \log X_0$$

where Y_i is

value	$\log(1-5f)$	$\log(1-2f)$	$\ldots$	$\log(1-10f)$
prob	$1/10$	$1/10$	$\ldots$	$1/10$

Payoffs uneven, complete information

Maximum $E(W_N)$

$$0 = \frac{-5}{1-5f} + \frac{-2}{1-2f} + \frac{-2}{1-f} + \frac{2}{1+f} + \frac{6}{1+2f} + \frac{10}{1+10f}$$

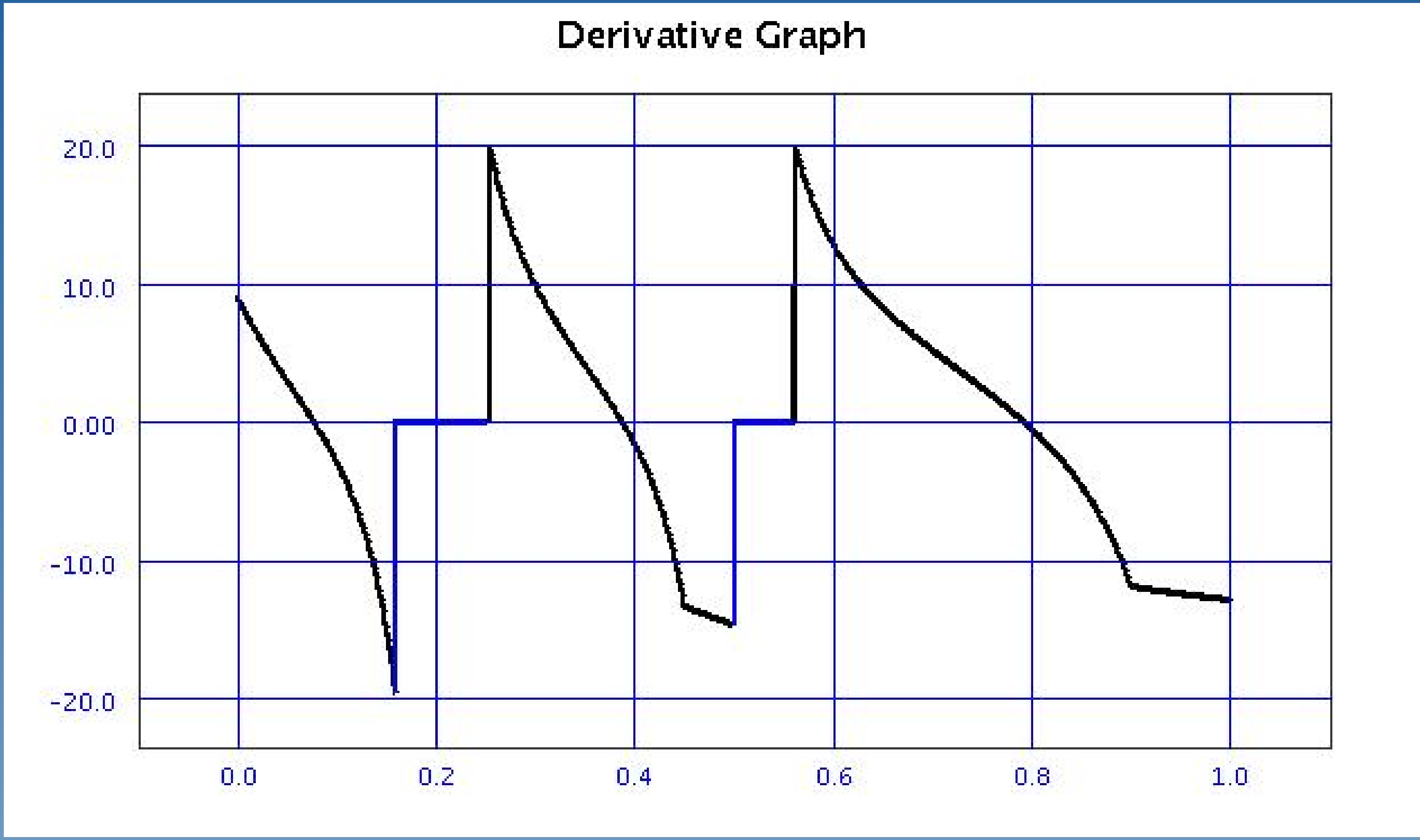
See next foil, solution is $f = 0.0774$ or $7\frac{3}{4}$%.

For this value of f, $E(Y) = 0.0335$ so our fortune grows like

$$X_N \approx X_0 e^{0.0335N} = X_0(1.0341)^N.$$

After 30 plays it should be $2.73X_0$ or \$273,000.

Payoffs uneven, complete information



Payoffs uneven, complete information

The general formula is: solve for the smallest f in

$$0 = \sum_i \frac{p_i r_i}{1 + r_i f}$$

where p_i is the probability of event i, r_i is the rate of return (positive)(or loss (negative) for event i on the amount at risk (equal to fX the fraction of the fortune).